Notes on Quantum Error Correction

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These notes draw primarily from Nielsen & Chuang, with various additions from papers and reviews published over the years.

1 Quantum Noise and the Operations Formalism

1.1 Overview

Quantum information processing systems are interesting to look at as closed systems, and we have built beautiful theoretical frameworks to examine them that way. However, this closed systems perspective can only ever be an approximation to the real thing. Real life systems suffer from unwanted interactions with their environments, which manifest in the form of noise. It is a very important task to describe, control, and ultimately minimise the effects of this noise on our systems. The *quantum operations formalism* is a useful tool in the quest to describe quantum noise and the behaviour of open quantum systems. A key factor behind this utility is that the formalism is applicable to a wide range of systems and scenarios; for systems that are either weakly or strongly coupled to their environments, or even systems that are initially closed and then suddenly become open. There are other nice things about it that we will discuss later.

The general structure will conform to that of Chapter 8 in Nielsen & Chuang. Understanding quantum error correction requires us to understand what quantum noise is, since it is this noise that produces the errors in quantum computers that these codes correct. Hence, we first go through Chapter 8 and then arrive at Chapter 10, hopefully armed with a thorough understanding of noise and a mathematical arsenal to implement.

1.2 Classical Noise

As we have done frequently in our explorations of quantum theory, it is often helpful to understand the classical before trying to take on the quantum. How does noise manifest in classical systems? Let's look at a simple example.

Example (Bit flips)**.** *Say we have a bit stored in a hard disk plugged into a classical computer. The bit is either in a state of 1 or 0 and over time, it's possible that magnetic fields from the environment cause the bit to flip value. Let the probability that the bit flips be p (and so the probability of it not flipping is $1-p$). To find what value p could have, we need to (1) model the distribution of magnetic fields in the environment – this could be accomplished by sampling the environmental magnetic field in environments similar to that which the hard drive finds itself in, and (2) model the interaction of the magnetic fields with the bits on the disk – that's easily described by Maxwell's equations, since in modern semiconductor memories, the bit's two states are simply two distinct levels of charge stored in a capacitor. Now, armed with a knowledge of the environment and the environment-system interaction, let's say the probability that the bit is initially in state 0 is p_0 and the probability that it is initially in state 1 is p_1. After the noise has influenced the system, say that the respective probabilities become q_0 and q_1. Let X be the* *initial state, and Y the final state. The probability $P(Y)$ is then*

$$P(Y = y) = \sum_x P(Y = y|X = x)P(X = x) \tag{1}$$

where the conditional probabilities $P(Y = y|X = x)$ are known as the transitional probabilities, summarising the changes due to the noise in the system. Writing this out as a matrix equation,

$$\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} (1-p)p_0 + pp_1 \\ pp_0 + (1-p)p_1 \end{bmatrix}. \tag{2}$$

A second, more complicated example follows:

Example (Markov processes)**.** *Say we're building a circuit to perform a computational task, but with faulty components. The circuit consists of two consecutive faulty NOT gates*[1]*, through which we put an initial bit X, an intermediate bit Y post the first NOT gate, and a final bit Z post the second NOT gate. It is a reasonable assumption that the correct or incorrect functioning of the second gate is independent of whether or not the first gate functioned correctly, i.e. the second noise process acts independently of the first. This process is then considered a Markov process, and is called stochastic.*

Physically, the assumption of Markovicity means that we assume the environment source of the noise for the first gate is independent of the environment source of the noise for the second – which makes sense because the gates are probably sufficiently far apart in space. We can describe noise in classical systems using stochastic processes, and in the case of multi-stage processes, we can often assume Markovicity. We can generalise Equation (1) to

$$\vec{q} = E\vec{p} \tag{3}$$

where E is a matrix of transition probabilities we can call the 'evolution matrix', $\vec{q}$ is the vector of final state probablities, and $\vec{p}$ is the vector of initial state probabilities This equation tells us that the initial state and the final state are linearly related. Describing quantum noise requires us to replace classical probability distributions with density matrices.

Remark (Properties of the evolution matrix)**.** *If $\vec{p}$ is a probability distribution, we require $E\vec{p}$ to also be a probability distribution. To satisfy this, there are two conditions we can impose on the entries of E:*

1. *(Positivity)* $[E]_{ij} \geqslant 0$.
2. *(Completeness)* $\sum_j [E]_{ij} = 1$

In summary, classical noise has the following key features:

- A linear relationship between initial state probabilities and final state probabilities, characterized by a matrix of transition probabilities called the *evolution matrix* whose entries are constrained by the positivity and completeness conditions.
- Noise processes with multiple stages can be described as *stochastic processes* provided we can make the assumption of independent noise sources: a property called *Markovicity.*

[1]Gates that reverse the state of the bit.

1.3 The Quantum Operations Formalism

We saw that Markov processes describe stochastic changes to classical states that we can write as vectors of probabilities. We can similarly describe the evolution of quantum states (such as stochastic changes to quantum states) using the quantum operations formalism. The density matrix ρ is the quantum analog of the vector of probabilities. Quantum states transform as

$$\rho' = \mathcal{E}(\rho) \tag{4}$$

where the map $\mathcal{E}$ is a quantum operation. We develop three equivalent approaches to quantum operations; (a) physically motivated axioms; (b) operator-sum representation; and (c) system-environment coupling, that each come with their own advantages depending on the system we intend to apply the formalism on. The figure below summarises the advantages and disadvantages of each approach.

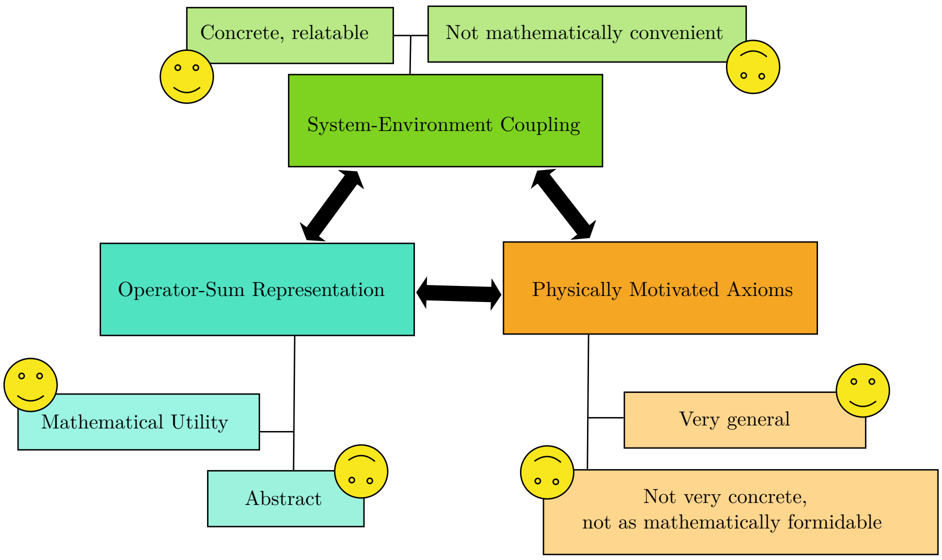

Figure 1: Three equivalent approaches to the quantum operations formalism, with their respective benefits and disadvantages.

1.3.1 Environments and Quantum Operations

A closed quantum system's dynamics are described by unitary transformations. For an open quantum system, we regard the dynamics as arising from the interaction between the system of interest and the environment, which taken together form a closed quantum system. Suppose we have a system in the state ρ sent into a box coupled to an environment. Generally, the final state of the system $\mathcal{E}(\rho)$ is not necessarily related to the initial state ρ by a unitary transformation. For now, it suffices[2] to assume that the input state of the system of interest - environment coupled system is a product state, $\rho \otimes \rho_{\text{env}}$. Post the transformation of the box by U, the system of interest no longer interacts with the environment and so we must 'trace over' the environment, i.e. take a partial trace with respect to the environment, to obtain a reduced density matrix that represents the final state of the system,

$$\mathcal{E}(\rho) = \text{tr}_{\text{env}}[U(\rho \otimes \rho_{\text{env}})U^\dagger]. \tag{5}$$

[2]This is not generally true. The generation of system-environment correlations makes this sort of separation via tensor product inaccurate. As thermodynamics tells us, if we leave a quantum system alone, its temperature eventually matches that of its environment, which generates these correlations. However(!), we generally prepare systems in specified states in a laboratory environment, where we destroy these correlations and have a pure state to work with. In this case, assuming the initial state is a product state is completely fine.

If the map U does not involve any interaction with the environment, then this trace procedure is unnecessary and we may simply just take

$$\mathcal{E}(\rho) = \tilde{U}\rho\tilde{U}^{\dagger} \tag{6}$$

where $\tilde{U}$ is the part of the map U that acts solely on the system. In other words, we can separate the unitary into parts that solely act on the system and the environment respectively and then write the final state of the system as the initial state transformed by the isolated system-acting unitary.

Equation (5) is the first of our three equivalent definitions of a quantum operation, and it comes from the system-environment coupling approach. Let us see how to use it in an enlightening example now.

Example (Two qubit circuit)**.** *Say we have a two qubit quantum circuit with the unitary operation being a controlled-NOT (CNOT) gate. The principal system is the control qubit, and environment is the target qubit. The environment is initially in the vacuum state, i.e.* $\rho_{env} = |0\rangle\langle 0|$. *The CNOT gate flips the target qubit if the control qubit is in the state* $|1\rangle\langle 1|$. *From (5), we have*

$$\mathcal{E}(\rho) = tr_{env}[U_{CNOT}(\rho \otimes |0\rangle\langle 0|)U^{\dagger}_{CNOT}] \tag{7}$$

Now, recall that the action of the CNOT gate on a product state in the computational basis is given by

$$U_{CNOT}\,|c\rangle\,|t\rangle = |c\rangle\,|t \oplus c\rangle \tag{8}$$

where $|c\rangle$ *is the control qubit and* $|t\rangle$ *the target qubit. Then, clearly*

$$U_{CNOT}(|0\rangle\,|0\rangle) = |0\rangle\,|0\rangle \tag{9}$$

$$U_{CNOT}(|1\rangle\,|0\rangle) = |1\rangle\,|1\rangle \tag{10}$$

i.e. the target qubit flips if and only if the control qubit is $|1\rangle$. *We can write the general density matrix* $\rho = \sum_{i,j=0}^{1} \rho_{ij}\,|i\rangle\langle j|$ *in the basis* $\{|0\rangle\,, |1\rangle\}$. *Then, the initial product state is*

$$\rho \otimes |0\rangle\langle 0| = \sum_{i,j=0}^{1} \rho_{ij}\,|i\rangle\langle j| \otimes |0\rangle\langle 0| \tag{11}$$

and acting the CNOT gate on each turn in the summation, we have

$$U_{CNOT}\,|i\rangle\,|0\rangle = |i\rangle\,|0 \oplus i\rangle = |i\rangle\,|i\rangle\,. \tag{12}$$

Hence,

$$U_{CNOT}(|i\rangle\langle j| \otimes |0\rangle\langle 0|)U^{\dagger}_{CNOT} = |i\rangle\langle j| \otimes |i\rangle\langle j|\,. \tag{13}$$

So,

$$U_{CNOT}(\rho \otimes |0\rangle\langle 0|)U^{\dagger}_{CNOT} = \sum_{i,j=0}^{1} \rho_{ij}\,|i\rangle\langle j| \otimes |i\rangle\langle j|\,. \tag{14}$$

Taking the partial trace over the second qubit (the vacuum), we have

$$tr_{vac}\left[\sum_{i,j=0}^{1} \rho_{ij}\,|i\rangle\langle j| \otimes |i\rangle\langle j|\right] = \sum_{i,j=0}^{1} \rho_{ij}\,|i\rangle\langle j|\, tr\,[|i\rangle\langle j|]\,. \tag{15}$$

However, $tr\,[|i\rangle\langle j|] = \langle j|i\rangle = \delta_{ij}$. *Only the terms with* $i = j$ *survive, yielding*

$$tr_{vac}\left[\sum_{i,j=0}^{1} \rho_{ij}\,|i\rangle\langle j| \otimes |i\rangle\langle j|\right] = \sum_{i,j=0}^{1} \rho_{ij}\,|i\rangle\langle j|\,\delta_{ij} = \rho_{00}\,|0\rangle\langle 0| + \rho_{11}\,|1\rangle\langle 1|\,. \tag{16}$$

Now, noting that $\rho_{00}\,|0\rangle\langle 0| = P_0\rho P_0$ *and* $\rho_{11}\,|1\rangle\langle 1| = P_1\rho P_1$ *where* P_0 *and* P_1 *are the projectors for the vacuum and* $|1\rangle$ *respectively, we can write the reduced state of the system as*

$$\mathcal{E}(\rho) = P_0\rho P_0 + P_1\rho P_1. \tag{17}$$

Why does this happen? Intuition tells us that this is the reduced state since the environment stays in the vacuum state only when the principal system is in the vacuum state. If this isn't the case, i.e. the principal system is in $|1\rangle$*, the environment flips to* $|1\rangle$ *as well.*